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QUALITATIVE ANALYSIS OF SYSTEMS WITH AN IDEAL NON-CONSERVATIVE CONSTRAINT*

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The Hamiltonian form developed in /1/ for the equations of motion of systems with ideal non-conservative constraints enables familiar methods of classical and celestial mechanics to be used to analyse the dynamics of such systems. When this is done certain difficulties arise, due to the fact that the Hamiltonian is not analytic. In this paper one of the possible algorithms applying KAM theory /2/ and Poincaré's theory of periodic motions /3/ to the analysis of systems in which the Hamiltonian is non-analytic in one of the phase variables is described. As an example, some results of /4/ concerning the dynamics of a rigid body colliding with a fixed, absolutely smooth, horizontal plane are refined.

1. Consider the motion of a nearly integrable Hamiltonian system with Hamiltonian

$$H = H_0(x_1, x_2, x_3) + \mu H_1(x_1, x_2, x_3, y_1, y_2, y_3, \mu) \quad (0 < \mu \ll 1) \tag{1.1}$$

The function (1.1) is 2π-periodic in the coordinates y_i $(i = 1, 2, 3)$, analytic in $\mu, x_1, x_2, x_3, y_1, y_2$, but only continuous with respect to y_3.

At $\mu = 0$ we have

$$x_i = x_{i0}, \quad y_i = \omega_i t + y_{i0} \quad (i = 1, 2, 3) \tag{1.2}$$

The zero subscript denotes the initial value of the relevant variable; the frequencies ω_i are equal to the derivatives $\partial H_0/\partial x_i$, evaluated at $x_j = x_{j0}$ $(j = 1, 2, 3)$.

Let us assume that the unperturbed motion (1.2) is conditionally periodic. Using Kolmogorov's theorem on conservation of motion /5, 6/, one can show that if H_0 satisfies certain restrictions, then, for sufficiently small μ and most initial values, the variables x_i will deviate only slightly (as long as μ is small) from their initial values for all t. Since the function (1.1) is not analytic, Kolmogorov's theorem cannot be applied directly, but one can use a version of the theorem adapted to symplectic maps /2, 7/.

To that end we introduce the variables p_i, where $x_i = x_{i0} + p_i$ $(i = 1, 2, 3)$. At the iso-energetic level,

$$H = H_0(x_{10}, x_{20}, x_{30}) + \mu h \quad (h = \text{const}, h \sim 1) \tag{1.3}$$

the motion can be described in terms of Whittaker's equations /8/. These equations are in Hamiltonian form with Hamiltonian K, where $p_3 = -K$ is a root of Eq.(1.3). The function K has the form

$$K = K_0(p_1, p_2) + \mu K_1(p_1, p_2, y_1, y_2, y_3, \mu, h) \tag{1.4}$$

where K_0 may be expressed as a series that converges for sufficiently small p_1, p_2:

$$K_0 = \omega_3^{-1}(\omega_1 p_1 + \omega_2 p_2) + {}^1/_2\omega_3^{-3}(a_{11}p_1^2 + 2a_{12}p_1p_2 + a_{22}p_2^2) + \ldots \tag{1.5}$$
$$a_{ii} = H_{0,ii}\omega_3^2 - 2H_{0,i3}\omega_i\omega_3 + H_{0,33}\omega_i^2 \quad (i = 1, 2)$$
$$a_{12} = H_{0,33}\omega_1\omega_2 + H_{0,12}\omega_3^2 - H_{0,13}\omega_2\omega_3 - H_{0,23}\omega_1\omega_3$$
$$(H_{0,ik} = \partial^2 H_0/\partial x_i \partial x_k)$$

The derivatives in (1.5) are evaluated at $x_i = x_{i0}$. The function K_1 in (1.4) is 2π-periodic in y_i $(i = 1, 2, 3)$, analytic in μ, p_i, y_i $(i = 1, 2)$ and continuous in y_3.

Let p_i', y_i' and p_i'', y_i'' be the values of p_i, y_i $(i = 1, 2)$ at $y_3 = 0$ and $y_3 = 2\pi$, respectively. By integration of the Whittaker equations

$$dy_i/dy_3 = \partial K/\partial p_i, \quad dp_i/dy_3 = -\partial K/\partial y_i \quad (i = 1, 2)$$

using series in powers of μ, we obtain a symplectic map $p_i', y_i' \to p_i'', y_i''$. This map is defined in terms of a generating function

$$S(p_1'', p_2'', y_1', y_2', \mu, h) = S_0 + \mu S_1 \tag{1.6}$$
$$S_0 = (y_1' + 2\pi\omega_1\omega_3^{-1})p_1'' + (y_2' + 2\pi\omega_2\omega_3^{-1})p_2'' + S_2^* + S_3^* + \ldots$$

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$$S_2^* = \pi\omega_3^{-3}(a_{11}p_1''^2 + 2a_{12}p_1''p_2'' + a_{22}p_2''^2)$$

(S_0 is a function analytic in the neighbourhood of the point $p_1'' = p_2'' = 0$, and S_k^* is a form of degree k in p_1'', p_2'').

If the condition

$$\delta \equiv \begin{vmatrix} \partial^2 S_0/\partial p_1''^2 & \partial^2 S_0/\partial p_1''\partial p_2'' \\ \partial^2 S_0/\partial p_1''\partial p_2'' & \partial^2 S_0/\partial p_2''^2 \end{vmatrix} \neq 0 \tag{1.7}$$

holds when $p_1'' = p_2'' = 0$ then the map $p_i', y_i' \to p_i'', y_i''$ is said to be non-degenerate. Assume that the non-degeneracy condition (1.7) is satisfied. Then /2, 7/ the map has two-dimensional invariant tori close to the "torus" $p_1' = p_2' = 0$ of the unperturbed (i.e., at $\mu = 0$) map; furthermore, the measure of the complement to the union of these tori is small together with μ. Hence, and by the equality $p_3 = -K$, it follows that than the variables x_1, x_2, x_3 differ only slightly from their initial values for most initial conditions and all t.

Performing some relatively easy algebra in (1.5)-(1.7), we can show that $\delta = -4\pi\omega_3^{-8}\Delta$, where

$$\Delta = \begin{vmatrix} H_{0,11} & H_{0,12} & H_{0,13} & \omega_1 \\ H_{0,12} & H_{0,22} & H_{0,23} & \omega_2 \\ H_{0,13} & H_{0,23} & H_{0,33} & \omega_3 \\ \omega_1 & \omega_2 & \omega_3 & 0 \end{vmatrix} \tag{1.8}$$

i.e., the map $p_i', y_i' \to p_i'', y_i''$ $(i = 1, 2)$ is non-degenerate if and only if the function H_0 is isoenergetically non-degenerate.

2. Consider a system with two degrees of freedom. The Hamiltonian

$$H = H_0(x_1, x_2) + \mu H_1(x_1, x_2, y_1, y_2, \mu) \quad (0 < \mu \ll 1) \tag{2.1}$$

is 2π-periodic in y_1, y_2, analytic in μ, x_1, x_2, y_1, but only continuous in y_2. Setting $x_i = x_{i0} + p_i$ $(i = 1, 2)$ and solving the equation $H = H_0(x_{10}, x_{20}) + \mu h$ for $p_2 = -K$, we arrive, as in Sect.1, at Whittaker's equations

$$dy_1/dy_2 = \partial K/\partial p_1, \quad dp_1/dy_2 = -\partial K/\partial y_1 \tag{2.2}$$

where $K = K(p_1, y_1, y_2, \mu, h)$ is 2π-periodic in y_1, y_2, analytic in μ, p_1, y_1, and its expansion in powers of μ is

$$K = K_0(p_1) + \mu K_1(p_1, y_1, y_2, h) + \ldots \tag{2.3}$$

Here $K_0(p_1)$ is a root of the equation $H_0(x_{10} + p_1, x_{20} - K_0) = H_0(x_{10}, x_{20})$,

$$\begin{gathered} K_0 = b_1 p_1 + b_2 p_1^2 + b_3 p_1^3 + \ldots \\ b_1 = \omega_1\omega_2^{-1}, \quad b_2 = {}^1/_2\omega_2^{-3}(H_{0,11}\omega_2^2 - 2H_{0,12}\omega_1\omega_2 + H_{0,22}\omega_1^2) \\ b_3 = {}^1/_6\omega_2^{-4}(H_{0,111}\omega_2^3 - 3H_{0,112}\omega_1\omega_2^2 + 3H_{0,122}\omega_1^2\omega_2 - \\ H_{0,222}\omega_1^3) + b_2\omega_2^{-2}(H_{0,22}\omega_1 - H_{0,12}\omega_2), \quad \omega_i = \partial H_0/\partial x_i \quad (i = 1, 2) \\ H_{0,ijk} = \partial^3 H_0/\partial x_i\partial x_j\partial x_k \end{gathered} \tag{2.4}$$

and the derivatives are evaluated at $x_i = x_{i0}$.

The function K_1 has the form

$$K_1 = -(\partial H_0/\partial x_2)^{-1}(H_1(x_1, x_2, y_1, y_2, 0) + h) \tag{2.5}$$

where the right-hand side is evaluated at $x_1 = x_{10} + p_1$, $x_2 = x_{20} - K_0$.

Eqs.(2.2) determine a symplectic map $p_1', y_1' \to p_2'', y_2''$,

$$\begin{gathered} y_1'' = y_1' + \gamma(p_1') + \mu f(p_1', y_1', \mu, h) \\ p_1'' = p_1' + \mu g(p_1', y_1', \mu, h) \end{gathered} \tag{2.6}$$

where p_1', y_1' and p_1'', y_1'' are the values of p_1, y_1 at $y_2 = 0$ and $y_2 = 2\pi$, respectively, the functions f and g are analytic in all their arguments, $\gamma = 2\pi\partial K_0(p_1')/\partial p_1'$.

The map (2.6) is area preserving. If at least one of the coefficients b_n $(n = 2, 3, \ldots)$ of the expansion (2.4) does not vanish, then for sufficiently small μ (2.6) is a twisting map and Moser's Invariant Curve Theorem /9/ is applicable. It follows from that theorem and the equality $p_2 = -K$ that, for sufficiently small μ and any initial conditions, the variables x_1, x_2 will differ only slightly from their initial values for all t.

3. If the ratio of the frequencies ω_1/ω_2 of the unperturbed system $(\mu = 0)$ with Hamiltonian (2.1) is a rational number m/n, then its motion

$$x_1 = x_{10},\ x_2 = x_{20},\ y_1 = \omega_1 t + \lambda,\ y_2 = \omega_2 t \tag{3.1}$$

is periodic with period $T_0 = 2\pi m\omega_1^{-1} = 2\pi n\omega_2^{-1}$. In (3.1) $\lambda = y_{10}$ and we have taken y_{20} equal to zero - this involves no loss of generality, since $\omega_2 \neq 0$ and the system is autonomous.

Suppose now that μ is small but not zero. The problem of the existence and stability of periodic solutions in a system with Hamiltonian (2.1) not analytic in y_2 can be solved by using the isoenergetic reduction of the equations governing the original system to Whittaker's Eqs.(2.2). System (2.2), in turn, can be tackled by means of Poincaré's algorithm for investigating periodic solutions (see /10/). Using this algorithm it can be shown that the system has a solution which is $2\pi n$-periodic with respect to y_2: $p_1 = p_1(y_2, \mu),\ y_1 = y_1(y_2, \mu)$. This solution is analytic in μ and reduces when $\mu = 0$ to the solution

$$p_1 = 0,\quad y_1 = (m/n)\, y_2 + \lambda$$

The functions

$$p_1 = p_1(y_2, \mu),\quad y_1 = y_1(y_2, \mu),\quad p_2 = -K \tag{3.2}$$

define a closed curve in the space of p_1, p_2, y_1, with y_2 treated as the curve parameter. The law of motion along the curve is determined by one of the equations of the original system

$$dy_2/dt = \partial H/\partial x_2$$

Substituting the function (3.2) into the right-hand side, we obtain

$$dy_2/dt = \omega_2 + \mu F(y_2, \mu, h) \tag{3.3}$$

where F is a $2\pi n$-periodic function of y_2, analytic in μ. Eq.(3.3) determines the time-dependence of y_2. It implies that the solution of the original equation corresponding to a solution of the reduced system which is $2\pi n$-periodic in y_2 is a T_μ-periodic function of t. The period

$$T_\mu = \int_0^{2\pi n} \frac{dy_2}{\omega_2 + \mu F} \tag{3.4}$$

is analytic with respect to μ and when $\mu = 0$ it equals the period T_0 of the unperturbed motion (3.1).

The algorithm of /10/ also yields conditions for periodic solutions of the reduced system to be stable in Lyapunov's sense. As applied to the original system, these conditions imply orbital stability of the T_μ-periodic solution.

Using results from /10/, one can prove the following assertion about periodic motions in systems with Hamiltonian (2.1).

Theorem. Let μ be sufficiently small and let $\langle H_1\rangle$ be the mean value of the function $H_1(x_1, x_2, y_1, y_2, 0)$ over the unperturbed motion (3.1), i.e.,

$$\langle H_1\rangle = \frac{1}{T_0}\int_0^{T_0} H_1(x_{10}, x_{20}, \omega_1 t + \lambda, \omega_2 t, 0)\, dt$$

Assume that the following conditions hold: 1) at $x_1 = x_{10},\ x_2 = x_{20}$ the function H_0 is isoenergetically non-degenerate, i.e., the coefficient b_2 in (2.4) does not vanish; 2) there exists λ_* such that at $\lambda = \lambda_*$

$$\partial\langle H_1\rangle/\partial\lambda = 0,\quad \partial^2\langle H_1\rangle/\partial\lambda^2 \neq 0.$$

Then the system with Hamiltonian (2.1) has a T_μ-periodic solution which is an analytic function of μ and reduces at $\mu = 0$ to a T_0-periodic solution (3.1) of the unperturbed system

$$x_1 = x_{10},\quad x_2 = x_{20},\quad y_1 = \omega_1 t + \lambda_*,\quad y_2 = \omega_2 t$$

If

$$b_2 \partial^2\langle H_1\rangle/\partial\lambda^2\, |_{\lambda=\lambda_*} > 0$$

this periodic motion is unstable, and if also

$$b_2 \frac{\partial^3\langle H_1\rangle}{\partial\lambda^3}\Big|_{\lambda=\lambda_*} < 0,\quad 5\left(\frac{\partial^3\langle H_1\rangle}{\partial\lambda^3}\right)^2 - 3\frac{\partial^2\langle H_1\rangle}{\partial\lambda^2}\frac{\partial^4\langle H_1\rangle}{\partial\lambda^4}\Big|_{\lambda=\lambda_*} \neq 0$$

then it is orbitally stable.

4. As an example, let us consider the motion of a rigid body performing collisions with a fixed, absolutely smooth, horizontal plane. We have already considered this problem /4/, but since the function (7.2) in /4/ is not analytic in W, the procedure used there to derive qualitative conclusions about the motion of the body needs improvement. We shall assume that the surface of the body is defined by an analytic function and differs only slightly (together with the small parameter μ) from a sphere of radius R centred at the centre of gravity of the body. The central ellipsoid of inertia is arbitrary.

The projection I_3 of the kinetic momentum of the body on the vertical is an integral, which we proceed to scale to the parameters of the problem. In the notation of /4/, the Hamiltonian can be written as

$$H = H_0 (I_1, I_2, I) + \mu H_1 (I_1, I_2, I, W_1, W_2, W) + \ldots \tag{4.1}$$

$$H_0 = H_0^{(1)} (I_1, I_2) + (9m\pi^2 g^2 I^2/32)^{1/3} \tag{4.2}$$

Here $H_0^{(1)}$ is the Hamiltonian for the motion of the body in the Euler-Poinsot case, I_2 is the kinetic momentum, m is the mass of the body, and g is the acceleration due to gravity. The quantity I is related to the height h to which the body rebounds from the plane in the unperturbed motion ($\mu = 0$) by the equality

$$I = {}^4/_3\pi^{-1} m\, (2gh^3)^{1/2}$$

In the unperturbed motion $I_i = I_{i0}$ $(i = 1, 2)$, $I = I_0$. We shall assume that $I_0 \neq 0$ (i.e., in the unperturbed motion the rebound height of the body is not zero), and the motion of the body relative to its centre of mass is conditionally periodic. The Hamiltonian (4.1) is 2π-periodic with respect to W_1, W_2, W and in a small neighbourhood of the unperturbed motion it is analytic in I_1, I_2, I, W_1, W_2; it is only continuous with respect to W.

We will now verify that H_0 is isoenergetically non-degenerate. Evaluating the determinant (1.8), we get

$$\Delta = -\delta_1 \partial\omega/\partial I - \delta_2 \omega^2 \tag{4.3}$$

$$\omega = \frac{2}{3}\left(\frac{9m\pi^2 g^2}{32I}\right)^{1/3} = \frac{\pi}{2}\left(\frac{g}{2h}\right)^{1/2}, \quad \delta_1 = \frac{\partial^2 H_0^{(1)}}{\partial I_1^2}\omega_2^2 - 2\frac{\partial^2 H_0^{(1)}}{\partial I_1 \partial I_2}\omega_1\omega_2 + \frac{\partial^2 H_0^{(1)}}{\partial I_2^2}\omega_1^2,$$

$$\delta_2 = \frac{\partial^2 H_0^{(1)}}{\partial I_1^2}\frac{\partial^2 H_0^{(1)}}{\partial I_2^2} - \left(\frac{\partial^2 H_0^{(1)}}{\partial I_1 \partial I_2}\right)^2 \tag{4.4}$$

Relying on computations from /11/, Chap. 2/, one can show that $\delta_1 = 2H_0^{(1)}\delta_2$. Using the expression for ω in (4.4), we obtain

$$\Delta = \omega^2 \delta_2 \,(H_0^{(1)}/(mgh) - 1) \tag{4.5}$$

It was shown in /11/ that $\delta_2 \neq 0$. Therefore, Δ may vanish only when $mgh = H_0^{(1)}$, i.e., $\Delta \not\equiv 0$, and the condition for the isoenergetic non-degeneracy of H_0 is satisfied. Consequently, as seen in Sect.1, for sufficiently small μ and most initial conditions the quantities I_1, I_2, I in the perturbed motion of the body remain close to their initial values for all t.

Suppose now that the body is dynamically and geometrically symmetric. Then the projection of the kinetic momentum on the axis of symmetry is an integral; it can be scaled to the parameters of the problem and investigation of the motion reduces /4/ to considering a system with two degrees of freedom:

$$H = H_0 + \mu H_1 (I_2, I, W_2, W) + \ldots \tag{4.6}$$

$$H_0 = {}^1/_2 I_2^2/A + (9m\pi^2 g^2/32)^{1/3} I^{2/3} \quad (A = {}^2/_5 mR^2)$$

The algorithm of Sect.2 can now be applied to the system with Hamiltonian (4.6). By (2.4), the coefficient b_2 is given by

$$b_2 = \frac{1}{2A\omega}\left(1 - \frac{I_2^2}{2Amgh}\right)$$

Thus b_2 can vanish only when $mgh = {}^1/_2 I_2^2/A$. But if $b_2 = 0$, then

$$b_3 = -\frac{I_2^3}{6A^3\omega^4}\frac{\partial^3 H_0}{\partial I^3} = -\frac{2I_2^3}{27A^3\omega^3 I^2} \neq 0$$

Consequently, by Sect.2, for sufficiently small μ and any initial conditions the quantities I_2, I in the perturbed motion will remain close to their initial values for all t.

In /4/ we considered periodic motions of a homogeneous ellipsoid of revolution, almost a sphere, colliding with a fixed absolutely smooth horizontal plane. In accordance with Sect.3, the discussion in /4/ should be augmented by adding the conditions: $mgh \neq {}^1/_2 I_2^2/A$ and the period is an analytic function of μ whose value at $\mu = 0$ is $\varkappa = 2\pi\omega^{-1} = 2\pi A I_2^{-1} k$ $(k = 1, 2 \ldots)$.

The existence and stability conditions for periodic solutions need no modification.

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ORBITAL STABILITY ANALYSIS USING FIRST INTEGRALS*

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A method is proposed for investigating the oribtal stability of periodic solutions of normal systems of ordinary differential equations. The Lyapunov function is derived from the first integrals of the equations of the perturbed motion and the scalar product of the velocity of motion along the orbit and the perturbation vector. Lypunov's second method was first used in connection with orbital stability in order to study the phase trajectories of systems with two degrees of freedom /1/.

1. ***Construction of the Lyapunov function.*** Let $\Omega \subset R^{n+1}$ be a domain containing the orbit /2/ of a T-periodic solution

$$Y = \Phi(t) \tag{1.1}$$

of the autonomous system

$$Y^{\cdot} = F(Y) \tag{1.2}$$

We shall investigate the orbital stability of (1.1) under the assumption that $F \in C^{(2)}(\Omega; R^{n+1})$.

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